CALCULATIONS OF PULSATILE FLOW ACROSS BIFURCATIONS IN DISTENSIBLE TUBES

W. A. HUNT

From the IBM Scientific Center, Houston, Texas 77025

ABSTRACT A simple method is given for extending across junctions the numerical methods previously used to study fluid flow in nonuniform, nonbranching, distensible tubes. Calculations with this method suggest that the ratio of downstream to upstream cross-sectional areas and the pressure wave velocity ratio in mammals are optimal for energy transfer across the junction.

INTRODUCTION

In recent years there have been many studies related to the flow of blood. Workers such as MacDonald (1) have combined measurements on laboratory animals with an investigation of the underlying physical phenomena. Other investigators have simulated blood flow by methods such as mathematical models and electrical analogs.

The more basic theoretical studies concern the behavior of an arterial segment, in the hope that if this is understood, some advance in understanding the arterial system can be expected. A comprehensive study in this direction was made in the 1950's by Womersley (2–5), who treated an arterial segment as a viscoelastic cylindrical tube containing viscous fluid. Womersley used a linear approximation to the Navier-Stokes equations and obtained analytical solutions for uniform tubes. Using a rubber tube model, Taylor (6) experimentally validated Womersley's approximations.

Streeter et al. (8) developed a nonlinear approach to the problem and solved their equations numerically using the method of characteristics. This approach was further developed by Barnard et al. (9–11) and applied to several interesting cases. With numerical methods, it is possible to treat the problem of nonuniform tubes and to impose a variety of boundary conditions.

The particular problem of wave reflections in the arterial system has been extensively studied. In 1952, Karreman (12) analyzed the reflection of a pressure wave at a sudden change in cross-section in an inviscid fluid flow and developed expressions for the reflection and transmission coefficients for this case. Womersley (5) devotes one chapter of his extensive report to the reflection phenomena, considering both the general case of branched tubing and the special case of the reflection from a cuff (short length of material of different elastic properties) surrounding the tube. Reflections are automatically included in the calculations of Streeter et al. (8) and Barnard et al. (9).

Martin (13) made theoretical and experimental studies of wave reflections in branched flexible conduits. In his experimental work, the effect of the branching angle was studied in the tubes with area branching ratio of 2.0 and 1.02. Martin found that the momentum changes (due to the bifurcation angle) do not make a significant contribution to the reflection characteristics.

Sawtell (14) studied experimentally the transmission of pulses in straight and bifurcating tubes. He varied the branching ratio in four steps from 1.09 to 1.80. The experimental points seem to confirm a minimum amount of reflected pressure amplitude between branching ratios of 1.09 and 1.21.

Sarpkaya (15) states that (*a*) the reflection coefficient is a minimum for area ratios between 1.1 and 1.3 for the range of Reynold numbers tested; (*b*) the effect of velocity ratio V/c on wave reflections is negligibly small for V/c between zero and 0.10; (*c*) the angle of divergence, (between 60 and 120 degrees) has very little or no effect on the reflection characteristics of otherwise identical bifurcations; and (*d*) the effect of the shape of the incident wave on the reflection coefficient is smaller than the scatter of the data.

Current models of the circulation system are usually networks in which the elements are arterial segments, treated by the linear approach (16). A solution is obtained for each of these segments and an electrical or digital analog obtained. From these linear models, impedances as a function of frequency can be calculated or analoged and the solutions for the frequency components summed to approximate a pulse wave.

It was of interest to extend the numerical calculations for nonuniform but nonbranching tubes to include the effects of bifurcations. This might then lead to a mathematical model of the circulatory system in which the nonuniformity of the arteries could be included. In this paper, the method for calculating across the bifurcation is discussed. The strategy followed is common in physics and engineering: the objective is limited to calculating the net effect of the junction, rather than the details of what happens at the bifurcation. This is accomplished by assuming that the bifurcation has zero length. The method of characteristics can then be extended in a simple way to calculate across the bifurcation.

Starting with the Navier-Stokes equations, the continuity equation, and an equation for a thin wall tube, Barnard et al. (9) developed a pair of quasi-linear hyperbolic partial differential equations in the two dependent variables pressure (p) and average velocity (U). The pair of equations derived were

$$\frac{\partial U}{\partial t} + \alpha U \frac{\partial U}{\partial z} + (1 - \alpha) U \frac{f'(p)}{f(p)} \frac{\partial p}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{\gamma \nu}{f(p)} \tag{1}$$

and

$$f'(p)\frac{\partial p}{\partial t} + f(p)\frac{\partial U}{\partial z} + Uf'(p)\frac{\partial p}{\partial z} = 0 \tag{2}$$

where $U = 1/A \int_0^R V_z\, 2\pi r\, dr$, V_z the instantaneous velocity of the fluid at a point (r, z),

$$\alpha = \frac{1}{AU^2}\int_0^R V_z^2 2\pi r\, dr$$

and γ is the frictional coefficient. The cross-sectional area (A) is defined as

$$A = f(p).$$

These equations were solved numerically using the method of characteristics. The characteristic curve in the z-t plane (Fig. 1) has the property that along it the partial differential equations 1 and 2 are equivalent to an ordinary differential equation of the form

$$\frac{dU}{dt} + g_2\frac{dp}{dt} = g_1 \tag{3}$$

where

$$g_2 = \frac{(1-\alpha)U}{p + kA_0} + \lambda$$

with

$$\lambda = \pm\left(\frac{1}{\rho(p + kA_0)} + \frac{\alpha(\alpha - 1)U^2}{(p + kA_0)^2}\right)^{1/2}$$

$$g_1 = -\frac{\gamma k U}{p + kA_0}$$

$$p = k(A - A_0)$$

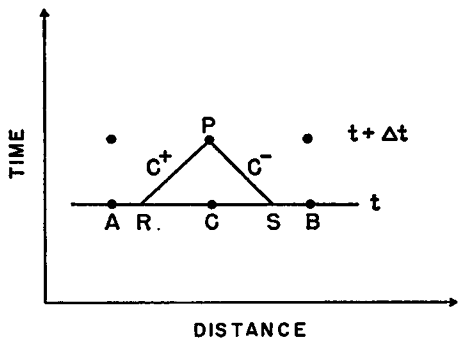

FIGURE 1

and A_0 = unstressed cross-sectional area. For a hyperbolic system there are two such curves passing through every point in the z-t plane, one for $\lambda+$ and one for $\lambda-$. For the point p, at time $(t + \Delta t)$ the curves are indicated by $C+$ and $C-$. Then for curve $C+$, equation 3 may be approximated by the finite-difference equation

$$(U_P - U_R) + (g_{2_c}^{+})(p_P - p_R) = g_{1_c}\Delta t \tag{4}$$

and for $C-$

$$(U_P - U_S) + (g_{2_c}^{-})(p_P - p_S) = g_{1_c}\Delta t \tag{5}$$

where the subscripts indicate the (z, t) points at which the quantities g_2^+, g_2^-, and g_1 are evaluated. Now if U and p are known at time t for the points, R, C, and S, the equation 4 and 5 may be solved simultaneously for the quantities U_P and p_P, which are values at time $(t + \Delta t)$. Thus the solutions may be propagated forward in time.

Consider now the boundary conditions. If either p or U is specified as a function of time for a boundary, then either equation 4 or 5 may be used to find the other. More generally, an equation specifying a pressure-flow relationship may be given at either end, and solved simultaneously with equations 4 or 5 to give p and U separately.

If the fluid and tube are assumed everywhere at rest at time zero, the problem is properly specified and may be solved numerically (18, 19). Density of the fluid ρ was set equal to 1.

EXTENSION ACROSS THE JUNCTION

To extend the method across bifurcations, a characteristic line is passed from point p in the time-distance plane into each of the tubes, shown schematically in Fig. 2.

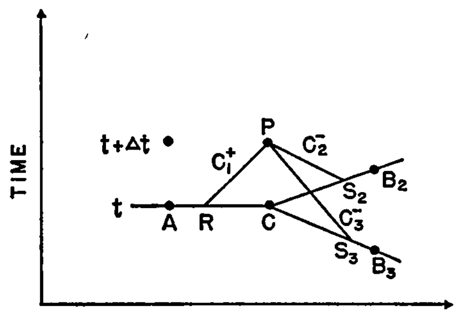

FIGURE 2

Three characteristic equations can be derived:

$$(U_{P_1} - U_R) + (g_{2_c}^{+})_1(p_P - p_R) = (g_{1_c})_1\Delta t \qquad (6)$$

$$(U_{P_2} - U_{S_2}) + (g_{2_c}^{-})_2(p_P - p_{S_2}) = (g_{1_c})_2\Delta t \qquad (7)$$

$$(U_{P_3} - U_{S_3}) + (g_{2_c}^{-})_3(p_P - p_{S_3}) = (g_{1_c})_3\Delta t. \qquad (8)$$

where U_{P_1}, U_{P_2}, U_{P_3}, are the velocities at the junction, point P, for tubes one, two, and three respectively. The conservation of fluid flow at the junction gives a fourth equation.

$$U_{P_1}A_{P_1} - U_{P_2}A_{P_2} - U_{P_3}A_{P_3} = 0$$

where A_{P_1}, A_{P_2}, A_{P_3} are the areas in tubes one, two and three respectively.

In Barnard et al. (9), an equation for a thin wall tube was derived which relates pressure to the area in the form of

$$P \simeq k(A - A_0)$$

solving for A and substituting in the continuity equation the following equation is obtained.

$$U_{P_1}\left(A_{01} + \frac{p_P}{k_1}\right) - U_{P_2}\left(A_{02} + \frac{p_P}{k_2}\right) - U_{P_3}\left(A_{03} + \frac{p_P}{k_3}\right) = 0 \qquad (9)$$

A quadratic equation is obtained by solving equations 6, 7, 8, and 9 for p_P. In order that p_P be bounded for a rigid tube, the positive sign for the radical is chosen. The value of U_{P_1}, U_{P_2}, and U_{P_3} can now be determined. For the remaining space points at $t + \Delta t$, tubes 1, 2, and 3 are then treated as separate tubes with the appropriate boundary conditions (9).

CALCULATIONS

To determine if the results were reasonable, a model was chosen consisting of an upstream tube jointed to two downstream tubes. The downstream tubes were chosen to be identical to reduce the amount of calculations. The parameters used for the upstream tube were the unstressed area of 2.37 cm² and k_1 of 4×10^6 dynes/cm². The cross-sectional area and stiffness of the upstream tube were chosen to agree with some experimental work by Sawtelle (14) who built some rubber bifurcating tubes with which he attempted to approximate the aorta in humans. The downstream tube parameters were calculated to give the desired area and pressure velocity ratios where by ratio is meant the downstream value divided by the upstream value. This model was programmed and calculated on an IBM 360/50.

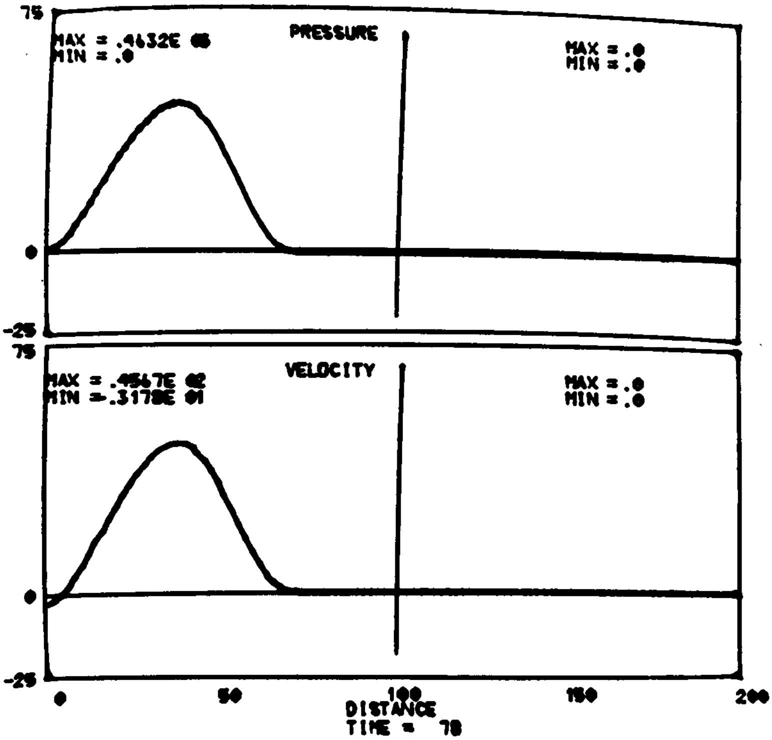

FIGURE 3a

The calculations were started by setting pressures and velocities at all space points to zero. The pressure pulse was introduced as a boundary condition at the left end of tube 1 (Fig. 2). When the desired pressure wave form had entered, the left end of tube 1 was terminated at the proximal end to look like an infinite tube. Tubes 2 and 3 were always terminated to look like infinite tubes. To further minimize reflection problems, the tubes were chosen to be 1000 cm long. Outputs from the calculations were pressure and velocity for each space mesh point for each time interval. These were plotted on an IBM 2250 Graphic Display Console and recorded on magnetic tape. A movie was made of the output. Fig. 3 is a sample of the output at various time steps. The ordinate represents the value of pressure or velocity, and the abscissa the distance along the tubes. Only one of the two tubes were plotted. The vertical line in the center shows the position of the bifurcation.

In the equations, there are some unknown parameters. These are the momentum coefficient, α, and the frictional coefficient, γ. These parameters are functions of the profile which must be assumed when using this method. It is assumed that the k values are measurable. Akers et al. (10), found that fluid resistance appears to be dependent on the driving frequency. Since values for these coefficients were not

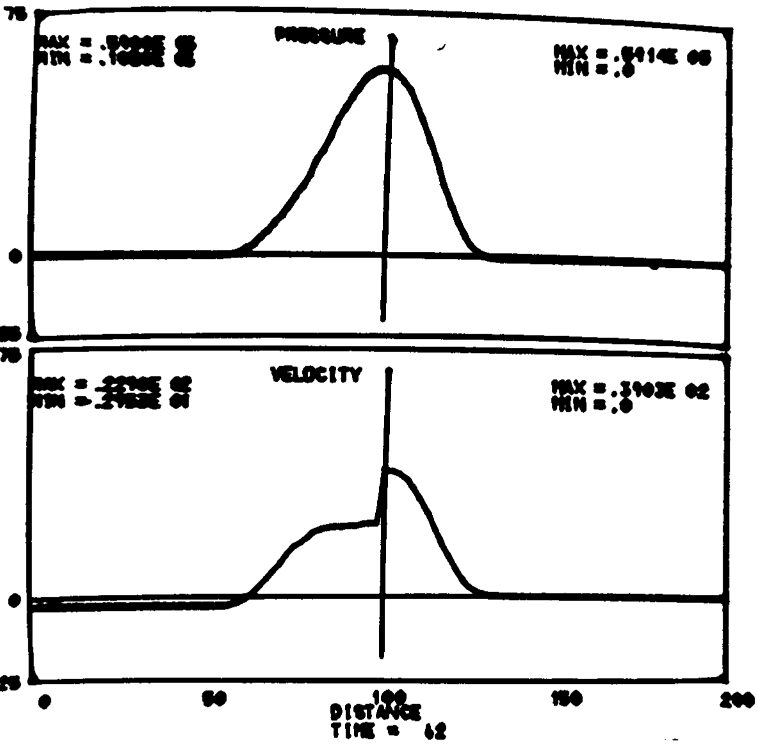

FIGURE 3b

known, they were varied in a systematic way to determine what effects this had upon the calculations.

For the first calculations, the pressure transmitted through the bifurcation normalized by the pressure for an equivalent tube without a bifurcation were plotted for a range of parameters (Figs. 4, 5, and 6).

To calculate the mechanical energy transfer across a plane perpendicular to the tube from time T_1 to time T_2, the equation used was

$$\sum_{i=T_1}^{T_2} (P_i + \tfrac{1}{2} \rho U_i^2) U_i A_i (t_i - t_{i-1}) \qquad (10)$$

The form of the input pressure wave was chosen to be

$$P_0(1 - \cos \theta)$$

with θ started at zero and terminated at 2π. The input was then terminated with its characteristic impedance. The mechanical energy which was transmitted past the

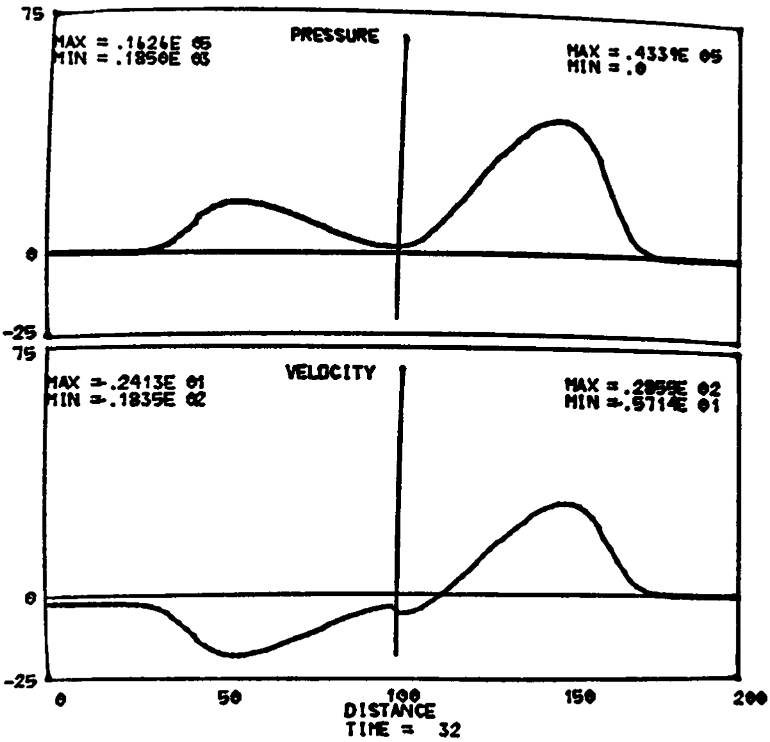

FIGURE 3c

bifurcation was summed and divided by the mechanical energy transmitted at the same space point in a tube with no bifurcation. The resulting data were plotted in Figs. 7, 8, and 9, where T is the periodic time in seconds for θ to vary from zero to 2π and c_0/c_i is the ratio of downstream to upstream pressure velocity.

The results were rather insensitive to the value of the parameter α. The variation with γ is shown on the figures.

An interesting aspect of the circulatory system is that the pressure wave velocity increases as the wave moves from the heart to the peripheral circulation. By varying k in the downstream tubes, the pressure wave velocity ratio between the downstream and upstream tubes for zero flow conditions can be controlled to give a constant pressure velocity ratio as the total area of the downstream tube is varied. Figs. 7, 8, and 9 show that as the pressure velocity ratio increases between downstream and upstream i.e. c_o/c_i, the optimum energy transfer across the bifurcation occurs at a higher area ratio.

All of the observable changes to the wave occurred in the 100 cm before the bifurcation. This indicates that in humans, many bifurcations and tube segments are interacting simultaneously on the wave.

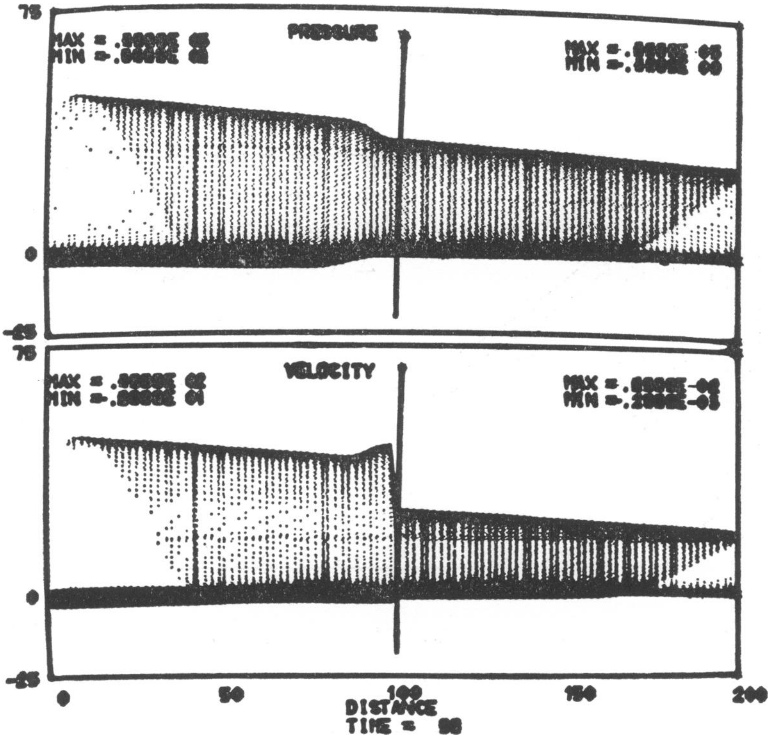

Figure 3d

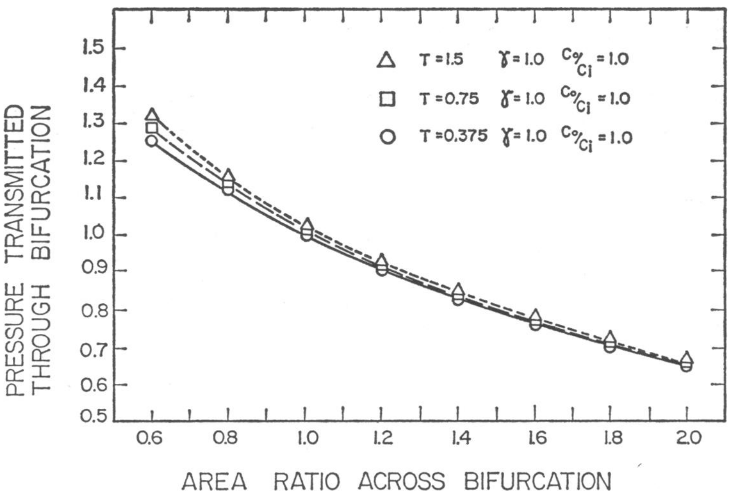

Figure 4

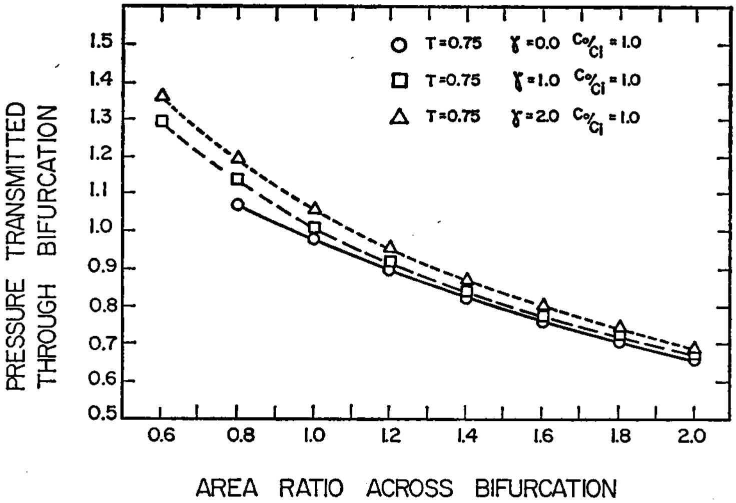

FIGURE 5

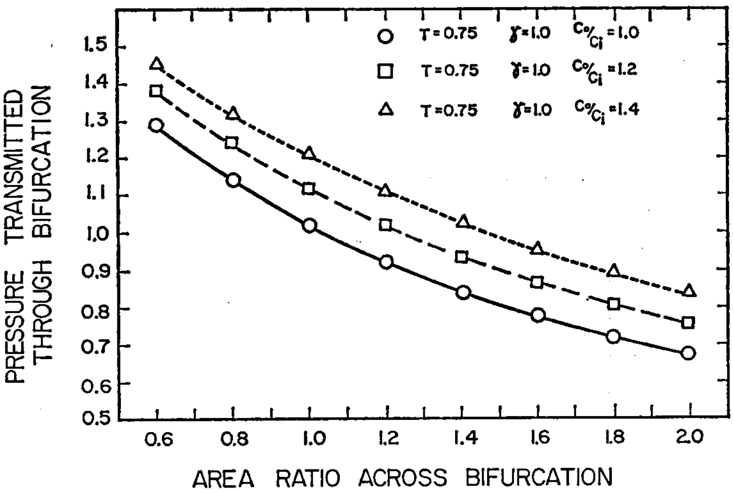

FIGURE 6

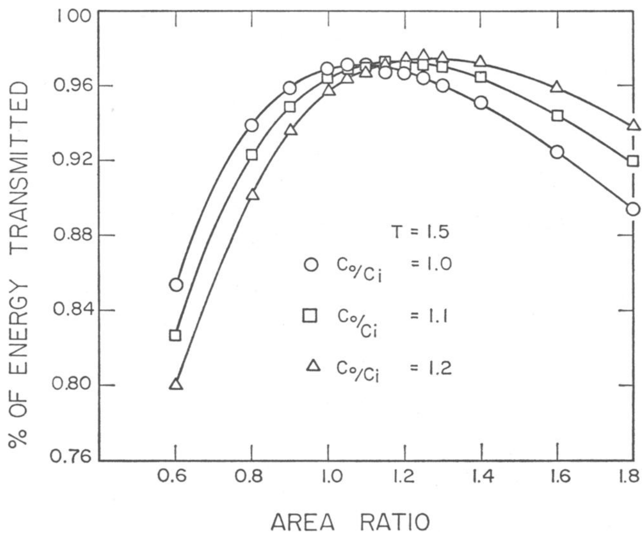

FIGURE 7

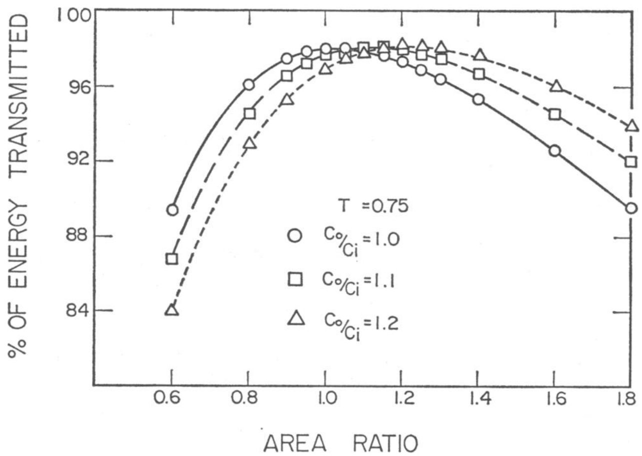

FIGURE 8

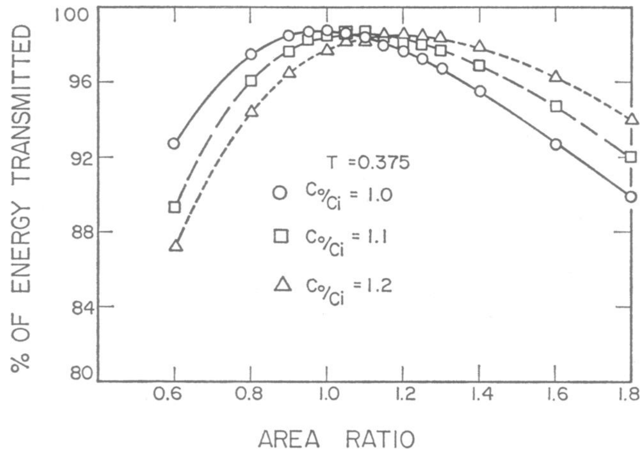

FIGURE 9

SUMMARY

This method for calculating across a junction together with the single tube method described by Barnard et al. (9) permits the modeling of the circulatory system with a set of equations which permit the use of nonuniform tubes.

The results of the calculations made to test this model suggest that the branching ratio in the circulatory system is near the optimum for energy transfer across a bifurcation.

Received for publication 14 February 1969.

REFERENCES

1. McDonald, D. A. 1960. Blood Flow in Arteries. Edward Arnold Ltd., London, England.
2. Womersley, J. R. 1955. *J. Physiol.* **127**:553.
3. Womersley, J. R. 1958. *Phys. Med. Biol.* **2**:178.
4. Womersley, J. R. 1958. *Phys. Med. Biol.* **2**:313, 374.
5. Womersley, J. R. 1957. An Elastic Tube Theory of Pulse Transmission and Oscillatory Flow in Mammalian Arteries. WADC Report TR 56–614.
6. Taylor, M. G. 1957. *Phys. Med. Biol.* **1**:258, 321.
7. Taylor, M. G. 1959. *Phys. Med. Biol.* **4**:63.
8. Streeter, V. L., W. F. Keitzer, and D. F. Bohr. 1964. *In* Pulsatile Blood Flow. E. O. Attinger, editor. 1964, McGraw-Hill Book Company, New York.
9. Barnard, A. C. L., W. A. Hunt, W. P. Timlake, and E. Varley. 1966. *Biophysical J.* **6**:717.
10. Akers, W. W., A. C. L. Barnard, H. M. Bourland, W. A. Hunt, W. P. Timlake, and E. Varley. 1966. *Biophysical J.* **6**:725.
11. Barnard, A. C. L., W. A. Hunt, W. P. Timlake, and E. Varley. 1966. *Biophysical J.* **6**:717.
12 Karreman, G. 1952. *Bull. Math. Biophys.* **14**:327.
13. Martin, J. D. 1966. Theoretical and experimental analysis of wave reflections in branched flexible conduits. Ph.D. Thesis. University of Illinois, Urbana, Ill.
14. Sawtelle, G. R. Transmission of pulse waves in straight and bifurcating elastic conduits. M. S. Thesis. Department of Mechanical Engineering, University of Nebraska, Lincoln, Neb.
15. Sarpkaya, T. 1967. Reflection and Transmission of Pulse Waves in Distensible Bifurcating Ves-

sels. Digest of 7th International Conference on Medical and Biological Engineering. Stockholm, Sweden.
16. NOODERGRAFF, A. 1964. *In* Pulsatile Blood Flow. McGraw-Hill, New York. Chapter 22.
17. ATTINGER, E. O. 1964. Pulsatile Blood Flow. McGraw-Hill, New York.
18. LISTER, M. 1964. *In* Mathematical Methods for Digital Computers. Ralston and Wilf, editors. John Wiley & Sons, Inc., New York.
19. FORSYTHE, G. E., and W. R. WASOW. Finite Difference Methods for Partial Differential Equations. John Wiley & Sons, Inc., New York.